Free Vibration Calculations of an Euler-Bernoulli Beam on an Elastic Foundation using He's Variational Iteration Method

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Abstract. Free vibration characteristics for an Euler-Bernoulli beam supported using a simple Winkler linear elastic foundation are calculated. The method of solution is by He's variational iteration method developed for various boundary (end) conditions. The beam's natural frequencies and mode shapes are obtained, with rapid convergence noted during the calculations. The calculation method is tested using a clamped-clamped beam. In this paper a robust and efficient algorithm is also given, based on the He's method, which can be easily modified for more complicated elastic foundations.

Keywords: Free vibrations, He's variational method, Winkler foundation.

1 Introduction

Beams resting on elastic foundations have a wide application in modern engineering, including railway engineering, but pose technical problems in structural design [1]. The Winkler model of elastic foundation is one of the simplest, where the vertical displacement of the beam is assumed to be proportional to the contact pressure at an arbitrary point [2]. A variety of investigations on free vibration, buckling and stability behavior of Winkler foundation beams have been conducted by researchers [3-6]. He's variational iteration method is a modification of a general Lagrange multiplier method [7] and has been used as a powerful tool for calculating free vibration [8-9]. In this paper, we proceed to investigate the free vibrations of an Euler-Bernoulli beam resting on an elastic foundation using the relatively new and more efficient method by He [10].

2 Governing Equation and Application of He's Method

The equation of motion for transverse vibrations of a uniform Euler-Bernoulli beam resting on a Winkler elastic foundation, as shown on Fig. 1, can be written as

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 w(x,t)}{\partial x^2}\right) + k_w(x)w(x,t) + \rho A(x)\left(\frac{\partial^2 w(x,t)}{\partial t^2}\right) = 0, \quad 0 < x < l. \tag{1}$$

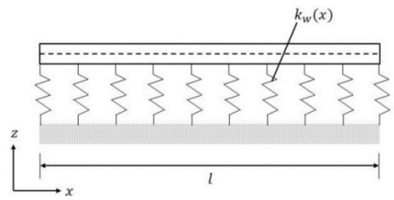


Fig. 1. Euler-Bernoulli beam on Winkler foundation

Using $w(x,t) = W(x)h(t)$ and without loss of generality Eq. (1) is now made non-dimensional to give

$$\frac{\mathrm{d}^4 W(X)}{\mathrm{d}X^4} - PW(X) = 0, \quad o < X < 1, \tag{2}$$

where P is the eigenvalue of the problem and is equal to

$$P = \frac{(k_w(X) - \rho A\omega^2)l^4}{EI}. \tag{3}$$

The correctional function can now be obtained

$$W_{n+1}(X) = W_n(X) + \int_0^X \lambda\left(\frac{\mathrm{d}^4 W_n(t)}{\mathrm{d}t^4} - PW_n(t)\right)\mathrm{d}t, \tag{4}$$

and the Lagrange multiplier λ can be found as

$$\lambda = \frac{(t-X)^3}{6}. \tag{5}$$

To start the iterations associated with Eq. (4) the $W_0(X)$ term is needed, which is represented as a Maclaurin series of the first four terms, and the solution of is found as $W(X) = \lim_{k\to\infty} W(X)_k$. The boundary conditions can also be written in dimensionless form

$$\left[\alpha_{r3}\frac{\mathrm{d}^3 W(X)}{\mathrm{d}X^3} + \alpha_{r2}\frac{\mathrm{d}^2 W(X)}{\mathrm{d}X^2} + \alpha_{r1}\frac{\mathrm{d}W(X)}{\mathrm{d}X} + \alpha_{r0}W(X)\right]\Bigg|_{X=0} = 0, \quad r = 1,2,$$

$$\left[\beta_{r3}\frac{\mathrm{d}^3 W(X)}{\mathrm{d}X^3} + \beta_{r2}\frac{\mathrm{d}^2 W(X)}{\mathrm{d}X^2} + \beta_{r1}\frac{\mathrm{d}W(X)}{\mathrm{d}X} + \beta_{r0}W(X)\right]\Bigg|_{X=1} = 0, \quad r = 1,2. \tag{6}$$

For Eq. (6), the second of the boundary conditions can be rewritten as

$$\sum_{j=0}^{3} f_{rj}^{[k]}(P)W^{(j)}(0) = 0, \quad r = 1,2. \tag{7}$$

Here, $f_{rj}^{|k|}$ are polynomials of P with respect to k. On solving the first boundary condition of Eq. (6) and Eq. (7) simultaneously for the non-trivial solutions $W^{(j)}(0)(j = 0,1,2,3)$, the ith eigenvalue $P_i^{[k]}$ corresponding to k can be obtained, and the number of iterations M is decided from

$$\left|P_i^{[n]} - P_i^{[n-1]}\right| \leq \varepsilon \tag{8}$$

3 Numerical Example: Clamped-Clamped (C-C) Beam

The following are results for a C-C beam. Important to this study is the efficiency of the method, which is demonstrated in Fig. 2. The variational iteration method proved extremely fast with convergence achieved after very few iterations. The linear modulus used in Fig. 2 and Table 1 took the form $k_w = k_{w0}(1 - \alpha X)$.

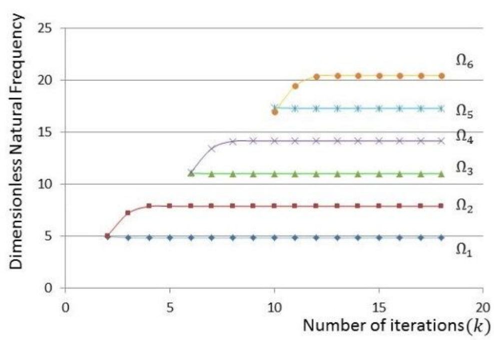
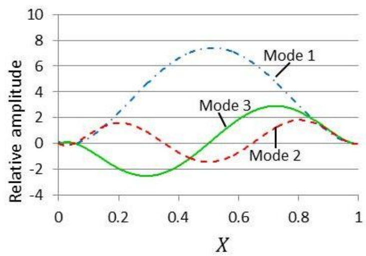


Fig. 2. Convergence for the first six natural frequencies when $k_{w0} = 50$, $\alpha = 0.2$.

The frequency parameters for the C-C beam with a linear elastic modulus are shown in Table 1. These values compare well with the equivalent found in the literature [9].

Table 1. Frequency parameters for C-C beam with linear foundation modulus ($\boldsymbol{\alpha} = \mathbf{0.2}$).

$\boldsymbol{k_{w0}}$	$\boldsymbol{\Omega_1}$	$\boldsymbol{\Omega_2}$	$\boldsymbol{\Omega_3}$	$\boldsymbol{\Omega_4}$	$\boldsymbol{\Omega_5}$	$\boldsymbol{\Omega_6}$
1	4.73217	7.85367	10.99578	14.13718	17.27933	20.39929
10	4.75116	7.85785	10.99730	14.13792	17.28016	20.39638
50	4.83294	7.87633	11.00407	14.14103	17.28250	20.41832
100	4.92965	7.89925	11.01250	14.14508	17.28361	20.42077
200	5.10758	7.94451	11.02930	14.15305	17.28789	20.39431
2000	6.92482	8.65221	11.31955	14.29380	17.36672	20.46750



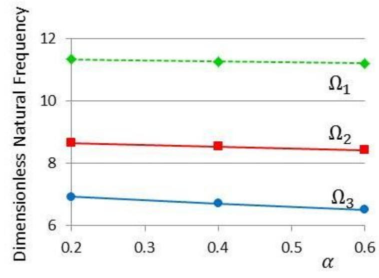


Fig. 3. Mode shapes for the first three natural vibrations and the effect of the linear foundation parameter, α on the natural frequencies.

The mode shape functions shown in Fig. 3 are obtained using the eigenvalues and a polynomial formed in terms of X. The second diagram in Fig. 3 shows the effect of increasing the slope of the linear function on the natural frequency. Basically the natural frequency falls linearly.

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